Math Virtual Learning

Calculus AB

Optimization using Derivatives

May 7, 2020

Calculus AB

Lesson: May 7, 2020

Objective/Learning Target:

Lesson 4 Derivatives Review

Students will complete optimization problems using derivatives.

Warm-Up:

Note: This is a review. For more examples refer back to your notes.

Watch Videos: Optimization part 1

Optimization part 2

Notes:

Process:

1. Determine that this is an **optimization problem** (wording includes words like "largest," "smallest," "greatest," "least," etc.).
2. **Draw a picture** (as always when working with word problems)
3. **Identify** what is **known and unknown**, and **assign variables** to the unknown quantities.
4. **Determine what value needs to be optimized** (maximized or minimized).
5. **Find a function** that models the value to be maximized or minimized.
6. If this function has two variables, use additional information in the problem to **eliminate one of the variables**. To do this, find a relationship between the two variables, usually given by some constraint in the problem, and solve for one variable in terms of the other.
7. **Find the absolute extreme value(s)** of the function of one variable, as you have learned to do previously.
8. **Determine the answer** by re-reading the question, and using your extreme value(s) found to find the answer.
9. **Check that your answer makes sense.**

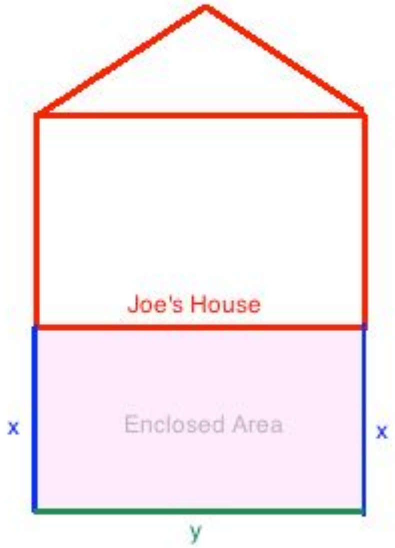
Example:

Joe wants to build a rectangular fence attached to his house. Joe has $400m$ of material, and wants the area enclosed in the fence to be a maximum. Find the dimensions of the field enclosed by the fence.

We first identify that we are looking for the maximum area enclosed by the fence, and that we have precisely $400m$ of material (our constraint) to work with. Let x be the two sides of the fence attached to Joe's house, and let y be the other side of the fence. The diagram below illustrates the problem:

We therefore get that we want to maximize the area $A = xy$, while we have a constraint that $2x + y = 400$. Now let's rewrite our constrait to be $y = 400 - 2x$, and thus:

$$A = xy$$
$$A = x(400 - 2x)$$
$$A = 400x - 2x^2$$
$$A' = 400 - 4x$$



If we let $A' = 0 = 400 - 4x$, we get that $x = 100$.

Therefore, when $x = 100$, $y = 200$, and the maximum area of the field will be $20000m^2$.

Examples:

2) A metal box (without a top) is to be constructed from a square sheet of metal that is 20 cm on a side by cutting square pieces of the same size from the corners of the sheet and then folding up the sides. Find the dimensions of the box with the largest volume that can be constructed in this manner.

$$V(x) = x(20-2x)(20-2x) = 400x - 80x^2 + 4x^3$$

$$V'(x) = 400 - 160x + 12x^2$$

$$400 - 160x + 12x^2 = 0 \Rightarrow 4(100 - 40x + 3x^2) = 0 \Rightarrow 4(3x-10)(x-10) \Rightarrow x = \frac{10}{3}, 10$$

$$V''(x) = -160 + 24x \quad V''\left(\frac{10}{3}\right) = -160 + 80 < 0 \quad V''(10) = -160 + 240 > 0$$

By the 2[nd] derivative test, the dimensions would be $\frac{10}{3}$ cm by $\frac{40}{3}$ cm by $\frac{40}{3}$ cm

Practice:

1) A rectangle has its vertices on the x-axis, on the y-axis, at the origin, and somewhere on the graph $y=4-x^2$ in the first quadrant. Find the maximum possible area of such a rectangle. Justify your answer.

2) A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$5 per meter, and the fencing for the other sides costs \$3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose.

Answer Key:

Once you have completed the problems, check your answers here.

1) **Solution:** As shown in the video, our rectangle has width x and height y, and so has area xy. But $y = 4 - x^2$, so our area is

$$A(x) = x(4 - x^2) = 4x - x^3.$$

This turns our word problem into just finding the maximum value of $A(x)$ on $[0, 2]$. Since $A'(x) = 4 - 3x^2$, we have a critical number when $4 - 3x^2 = 0$, or $x = \frac{2}{\sqrt{3}}$. Then $y = 4 - x^2 = 4 - \frac{4}{3} = \frac{8}{3}$, and the area is

$$A = xy = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}.$$

Video

Answer Key:

Once you have completed the problems, check your answers here.

2)

Call the length of fence along the river x, and the length perpendicular to the river y.

$C(x) = 5x + 3(2y + x) \qquad xy = 1200 \Rightarrow y = \frac{1200}{x} \Rightarrow C(x) = 8x + \frac{7200}{x}$

$C'(x) = 8 - \frac{7200}{x^2} \quad 8 - \frac{7200}{x^2} = 0 \Rightarrow 8x^2 = 7200 \Rightarrow x^2 = 900 \Rightarrow x = 30$

$C''(x) = \frac{14400}{x^3} \quad C''(30) = \frac{14400}{30^3} > 0$

By the 2nd derivative test, a field that is 30 m along the river by 40 m perpendicular to the river would be least expensive.

Additional Practice:

[Interactive Practice]

[Extra Practice with Answers]